Time evolution of the eddy viscosity in two-dimensional Navier-Stokes flow

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The time evolution of the eddy viscosity associated with an unforced two-dimensional incompressible Navier-Stokes flow is analyzed by direct numerical simulation. The initial condition is such that the eddy viscosity is isotropic and negative. It is shown by concrete examples that the Navier-Stokes dynamics stabilizes negative eddy viscosity effects. In other words, this dynamics moves monotonically the initial negative eddy viscosity to positive values before relaxation due to viscous term occurs.

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Molecular viscosity ν has its origin in the collisions between the particles of a fluid: if the flow is submitted to a shear, one obtains a momentum flux proportional (and with opposite sign) to the velocity gradient. The crucial hypothesis for obtaining *transport coefficients*, like the molecular viscosity, is the *scale separation* $\varepsilon_0 = \ell_0/\mathcal{L}_0 \ll 1$: macroscopic phenomena should occur on scales $\mathcal{L}_0$ much larger than the mean free path ℓ_0 of the particles of the fluid. By analogy, if one liquid is macroscopically selforganized in cellular motion (typical size of the cells $\sim\ell$), one gets the so-called *eddy viscosity*, which is used to describe the flow motion on very large scales $\mathcal{L} \gg \ell$. The basic cellular motion may have different sources, e.g. a small-scale instability. In the present case, we will assume that the basic flow is being maintained by an external force. The idea of eddy viscosity was developed on earlier times by Boussinesq [1] and his successors [2,3], for the modeling of (i) turbulence, (ii) several astrophysical phenomena, (iii) the K-ϵ industrial modeling, etc. However, only recently it was possible to develop a complete theory for the eddy viscosities, since certain main ingredients like the scale separation $\varepsilon = \ell/\mathcal{L}$ and the absence of AKA effects were absent in the previous phenomenological approaches [4].

The aim of this work is to follow the time evolution of the eddy viscosity of two-dimensional incompressible liquids. The dynamics of these fluids are governed by the well-known incompressible Navier-Stokes equations. When their solutions are analyzed on very large scales, their molecular viscosity ν is renormalized into a fourth-order eddy viscosity tensor $\nu_{ij\ell m}$ which, assuming that the eddy cellular motion, i.e., the basic flow possesses the symmetries $(S1)$ or parity-invariance, and $(S2)$ or sixfold rotation symmetry, is reduced to $\nu_{ij\ell m} = \nu_{\rm eddy}\delta_{j\ell}\delta_{im}$, where $\nu_{\rm eddy}$ is the usual eddy viscosity (isotropy of the fourth-order eddy viscosity tensor) [4]. If $\nu_{\rm eddy}$ is negative [5], then the linear dissipative term associated to the large scale dynamics (scales $\sim\mathcal{L}$) will induce a large scale instability. This situation is a rather common phenomenon: about 33% of the two-dimensional basic flows displaying the symmetries $(S1)$ and $(S2)$, with an initial energy spectrum $E(K) \sim K^{-3}$, lead to a negative eddy viscosity when lowering the molecular viscosity [6]. It is, therefore, relevant to find out what will happen to these flows (which, for the moment, merely describe an initial randomly chosen state) when they are subjected to the dynamics induced by the Navier-Stokes equations.

Briefly, let us summarize the mathematical machinery that allow us to compute eddy viscosities in two dimensions [5] [6]. Consider the two-dimensional unforced Navier-Stokes equation, which in a stream function $\Psi(t;x_1,x_2)$ formalism reads

$$\partial_t \partial^2 \Psi + J(\partial^2\Psi, \Psi) = \nu \partial^2 \partial^2 \Psi. \tag{1}$$

Here, J is the Jacobian, ∂^2 is the Laplacian and ν is the molecular viscosity. The velocity field $\mathbf{v} = (v_1, v_2)$ is defined as $v_i = \varepsilon_{ij}\partial_j \Psi$ (the tensor ε_{ij} is the antisymmetric tensor having $\varepsilon_{12} = -\varepsilon_{21} = 1$; zero, otherwise). Assume Ψ periodic in x_1 and x_2. Let $\Psi_t(x_1,x_2)$ be the solution of Eq. (1) initialized by the periodic initial condition $\Psi_0(x_1,x_2)$. Then, under the symmetries $(S1)$ and $(S2)$, the two-dimensional eddy viscosity $\nu_{\rm eddy}(\Psi_t,\nu)$, associated to the basic flow Ψ_t with molecular viscosity ν, is given by

$$\nu_{\rm eddy}(\Psi_t,\nu) = \nu - \langle Q_t(\partial_2\Psi_t)\rangle - 2\langle(\partial_1 S_t)(\partial_2\Psi_t)\rangle, \tag{2}$$

where $\langle\bullet\rangle$ denotes the average over the space periodicities and $\partial_i = \partial/\partial x_i$ $(i=1,2)$. The quantities Q_t and S_t are the solutions of the auxiliary problems:

$$\tilde{\mathcal{A}}_t Q_t = \partial_2 \partial^2 \Psi_t, \tag{3}$$

$$\tilde{\mathcal{A}}_t S_t = (\partial_2\partial^2\Psi_t)Q_t + 2J(\Psi_t, \partial_1 Q_t) - (\partial_2\Psi_t)(\partial^2 Q_t) + 4\nu\,\partial_1\partial^2 Q_t. \tag{4}$$

Here, $\tilde{\mathcal{A}}_t$ is the linearized Navier-Stokes operator:

$$\tilde{A}_t\psi \equiv J(\partial^2\psi, \Psi_t) + J(\partial^2\Psi_t, \psi) - \nu\partial^2\partial^2\psi, \tag{5}$$

restricted to functions of zero mean value, which have the same space-time periodicity as the basic flow Ψ_t.

Let us state our problem. Assume that we are given an initial stream function Ψ_0 and a correspondent molecular viscosity ν such that the associated eddy viscosity $\nu_{\rm eddy}(\Psi_0,\nu)$ is negative. We want to study the quantity

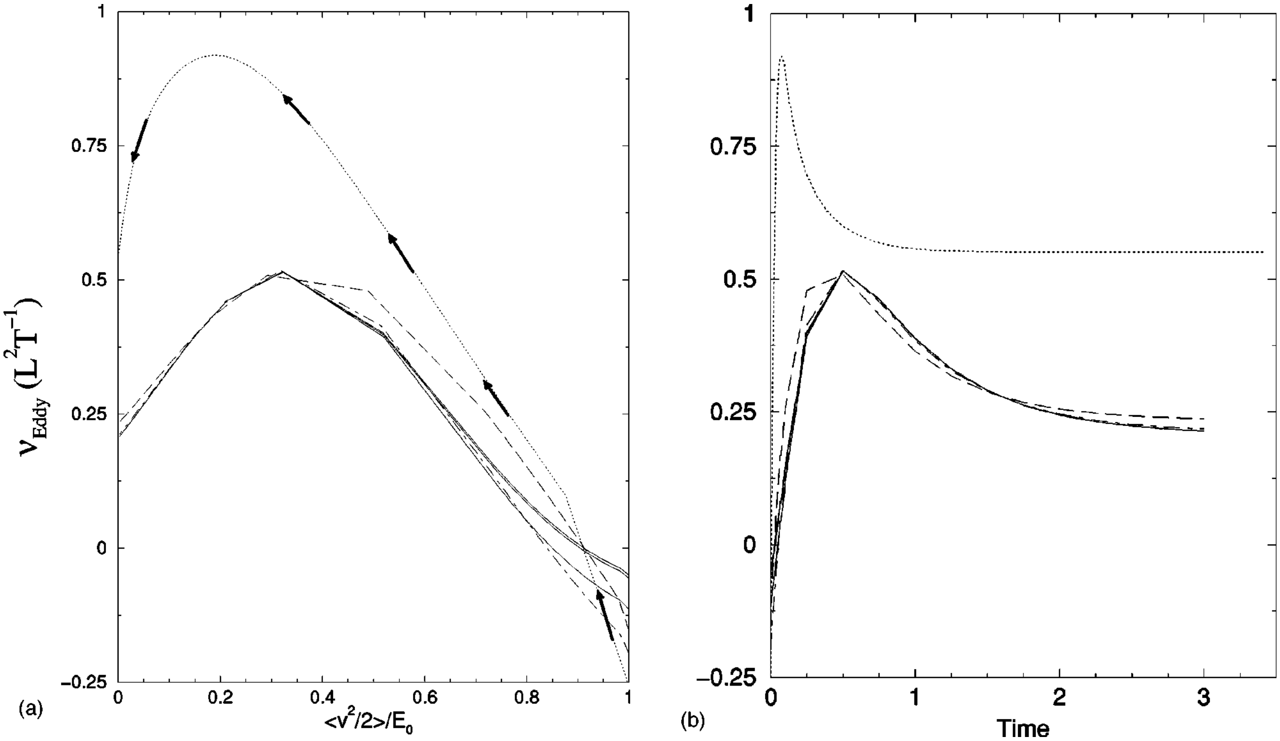

FIG. 1. Time evolution of the eddy viscosity $\nu_{\rm eddy}$ of six random flows (excepted where indicated) *versus* their (a) mean energy over its initial mean energy, and (b) time. It is clearly seen that the eddy viscosity instantly increases to positive values when the Navier-Stokes dynamics operates on the flows; dotted line: the decorated hexagonal flow introduced in Ref. [5], $\nu=0.55$; solid lines: 3 runs with $\nu=0.205$; dot-dashed line: $\nu=0.21$; long-dashed line: $\nu=0.23$.

$\nu_{\rm eddy}(\Psi_t,\nu)$, for $t>0$, in particular, to know what happens to its sign. In order to investigate this, we start by considering pairs (Ψ_0,ν) such that:

Ψ_0 satisfies $(S1)$ and $(S2)$; this is done by choosing different lengths for the two-dimensional periodicity box ($L_1=2\pi$ and $L_2=2\pi/\sqrt{3}$), and selecting wavevectors of the form $\mathbf{k}=(2\alpha+\beta)(1,0)+\beta\sqrt{3}(0,1)$, with α,β signed integers, such that the Fourier modes of Ψ_0, which can be obtained by rotations of $\pi/3$ have the same real Fourier amplitudes;

Ψ_0 has an energy spectrum $\sim K^{-3}$ $(1\leqslant K\leqslant 7$; zero, otherwise), with mean energy renormalized to unity;

$\nu_{\rm eddy}(\Psi_0,\nu)<0$.

To encounter such pairs several stratagies can be followed, in particular, the one mentioned in Ref. [6]. When one of these pairs is identified, we numerically compute Ψ_t, using Eq. (1) initialized with (Ψ_0,ν), and find the stationary solutions Q_t,S_t of the auxiliary problems (3) and (4) in order to calculate (2). To perform these computations, we use the standard pseudospectral method with N^2 Fourier modes, with deliasing by truncation beyond wave number $N/3$ [7], and the slaved-frog temporal scheme [8]. The results reported in this letter require only $N=64$ or $N=128$ and they are not sensitive if the resolution N is duplicated (the Reynolds numbers are ~ 10).

Several runs were performed under these conditions and all look similar. Figure 1 presents six of these runs. It is remarkable that in all the cases $\nu_{\rm eddy}(\Psi_t,\nu)$ always increases from the very beginning, i.e., $d\nu_{\rm eddy}(\Psi_t,\nu)/dt>0$, for $t\in[0,T^*]$, where T^* corresponds to the time where the eddy viscosity reaches its maximum value, which is positive. The time interval $[0,T^*]$ is the period of time where the Navier-Stokes dynamics is dominated by the nonlinearities, beyond which all the dynamics is relaxed by dissipative terms.

A simple observation shows that, since the Navier-Stokes Eq. (1) is unforced, as time goes by, we have $\Psi_t\to 0$, which implies $Q_t,S_t\to 0$, and thus $\nu_{\rm eddy}(\Psi_t,\nu)\to\nu$, i.e., $\nu_{\rm eddy}(\Psi_t,\nu)$ becomes always positive. This can be observed in Fig. 1(b). The increase of the eddy viscosity mentioned above is not due to the previous purely dissipative effect. Indeed, Fig. 1(a), plotting the eddy viscosity $\nu_{\rm eddy}$ *versus* the dimensionless quantity $\langle v^2/2\rangle/E_0$ (the mean energy of the fluid $\langle \frac{1}{2}v^2\rangle=\frac{1}{2}1/L_1 1/L_2\int_0^{L_1}\int_0^{L_2}v^2(t;x_1,x_2)\,dx_1dx_2$, over its initial mean energy $E_0=\langle\frac{1}{2}v^2(0;x_1,x_2)\rangle$, shows that the initial negative isotropic eddy viscosity immediately increases to positive values before relaxation due to viscous term occurs [9].

In summary, we have performed numerical experiments on the unforced two-dimensional Navier-Stokes equations initialized with initial conditions and a given molecular viscosity such that the corresponding initial eddy viscosity is negative. We have shown that the Navier-Stokes dynamics stabilizes negative isotropic geometrical eddy systems in the sense that these systems, when submitted to the Navier-Stokes dynamics, evolve to topological configurations characterized by positive isotropic eddy viscosities.

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[9] Figure 1(a) should be read following the indications given by the arrows, i.e., from the right to the left.